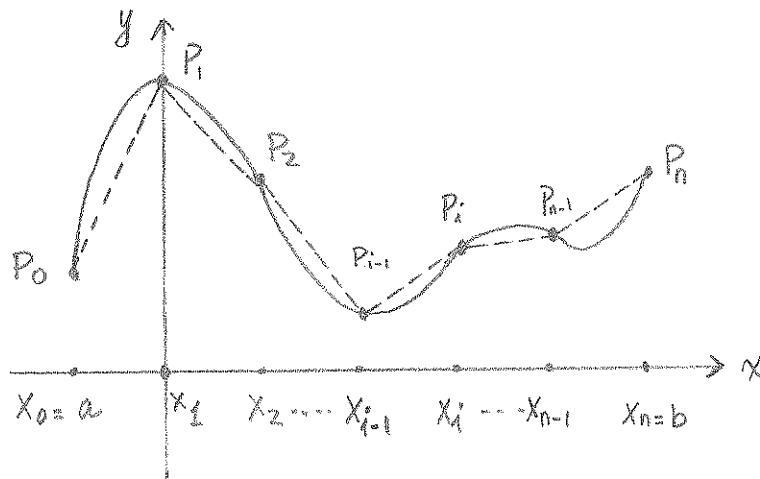


Chapter 8 - Further Applications of Integration

Section 8.1 - Arc Length

- Goal: Calculate the length of a curve.

Consider a continuous function $f(x)$ over an interval $[a, b]$, as shown below. Divide the interval $[a, b]$ into n subintervals of equal length Δx



so that $\Delta x = \frac{b-a}{n}$. Let x_0, x_1, \dots, x_n denote the endpoints of these intervals, and let P_i be the point on the graph of f corresponding to $(x_i, f(x_i))$.

Denote by $|P_{i-1}P_i|$ the length of the

line segment between P_{i-1} and P_i . Then an approximation to the length of the curve between $x=a$ and $x=b$ is $L \approx \sum_{i=1}^n |P_{i-1}P_i|$. To get the best approximation possible, we would need to make n very large ($n \rightarrow \infty$) or equivalently, make Δx very small ($\Delta x \rightarrow 0$). We calculate

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} \quad \text{where } \Delta y_i = y_i - y_{i-1}.$$

By the Mean Value Theorem, on each interval $[x_{i-1}, x_i]$, there exists a point x_i^* such that $f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x}$, or equivalently, $\Delta y_i = f'(x_i^*) \cdot \Delta x$. With this, we have

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (f'(x_i^*) \cdot \Delta x)^2} = \Delta x \sqrt{1 + [f'(x_i^*)]^2}. \quad \text{Therefore}$$

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n |P_{i-1}P_i| = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + [f'(x_i^*)]^2} \cdot \Delta x = \int_a^b \sqrt{1 + (f'(x))^2} dx, \quad \text{Since}$$

the sum above is a Riemann sum.

Arc length formula: if f' is continuous on $[a, b]$, then the length of $y = f(x)$, $a \leq x \leq b$ is $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$. If the curve is given as a function of y , i.e. $x = g(y)$, then the arc length between $y=c$ and $y=d$ is $L = \int_c^d \sqrt{1 + (g'(y))^2} dy$

Examples ① find the arc length of $y = 6x^{3/2}$ for $0 \leq x \leq 5$.

We have $f'(x) = 6 \cdot \frac{3}{2} x^{1/2} = 9\sqrt{x}$, so that $1 + (f'(x))^2 = 1 + 81x$. Thus,

$$L = \int_0^5 \sqrt{1 + 81x} dx = \frac{1}{81} \cdot \frac{2}{3} (1 + 81x)^{3/2} \Big|_0^5 = \frac{2}{243} (406\sqrt{406} - 1).$$

② find the arc length of $x = \frac{1}{2}y^4 + \frac{1}{16}y^2 - 3$ for $1 \leq y \leq 2$.

We have $g'(y) = \frac{dx}{dy} = 2y^3 - \frac{1}{8}y^{-3}$ so that $(g'(y))^2 = 4y^6 - \frac{1}{2} + \frac{1}{64}y^{-6}$.

Thus $1 + (g'(y))^2 = 4y^6 + \frac{1}{2} + \frac{1}{64}y^{-6} = (2y^3 + \frac{1}{8}y^{-3})^2$. Therefore,

$$L = \int_1^2 \sqrt{(2y^3 + \frac{1}{8}y^{-3})^2} dy = \int_1^2 2y^3 + \frac{1}{8}y^{-3} dy.$$

③ Prove that the Circumference of a circle of radius R is $L = 2\pi R$.

The upper semi-circle of radius R has equation $y = \sqrt{R^2 - x^2}$, whence

$\frac{dy}{dx} = \frac{-2x}{2\sqrt{R^2 - x^2}}$, and so $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}$. The circumference L is

$$L = 2 \cdot \int_{-R}^R \sqrt{\frac{R^2}{R^2 - x^2}} dx = 4 \cdot \int_0^R \frac{R}{\sqrt{R^2 - x^2}} dx. \text{ Let } x = R \sin \theta, dx = R \cos \theta d\theta.$$

$$x=0 \Rightarrow \theta=0, x=R \Rightarrow \theta=\frac{\pi}{2}.$$

$$\text{We have } L = 4 \int_0^{\frac{\pi}{2}} \frac{R \cdot R \cos \theta d\theta}{\sqrt{R^2 - R^2 \sin^2 \theta}} = 4 \int_0^{\frac{\pi}{2}} R \cdot \frac{R \cos \theta}{\sqrt{R^2 \cos^2 \theta}} d\theta = 4R\theta \Big|_0^{\frac{\pi}{2}} = \boxed{2\pi R}.$$